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p-SATURATED FORMATIONS

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ABSTRACT

We consider here a nonempty formation $\mathfrak{F}$ locally defined by a system $\{\mathfrak{F}(t)\}$ where for some fixed prime number p, $\mathfrak{F}(p)=\mathfrak{F}$. We call such formations p-locally defined. It is shown that every p-locally defined formation $\mathfrak{F}$ has the property that $G/\Phi_p(G)\in\mathfrak{F}$ implies that $G\in\mathfrak{F}$ where $\Phi_p(G)$ is the p-Frattini subgroup of G. If $\mathfrak{F}$ is a formation of solvable groups, then this property is equivalent to $\mathfrak{F}$ having a p-local definition. For solvable groups equivalent conditions are stated in terms of the $\mathfrak{F}$-projectors, the $\mathfrak{F}$-normalizers, and the $\mathfrak{F}$-hypercenter.

Since W. Gaschutz defined them in [4], there has been considerable discussion about saturated formations and locally defined formations. We consider here a nonempty formation $\mathfrak{F}$ locally defined by a system $\{\mathfrak{F}(t)\}$ where, for some fixed prime number p, $\mathfrak{F}(p)=\mathfrak{F}$. In other words, the defining formation corresponding to the prime number p will be restricted as little as possible. As it turns out, such formations have an interesting relation to the p-Frattini subgroup as defined by W. E. Deskins [2]. If we restrict our attention to formations of solvable groups locally defined by an integrated system $\{\mathfrak{F}(t)\}$ with $\mathfrak{F}(p)=\mathfrak{F}$, then it is obvious that a solvable group G can have no $\mathfrak{F}$-crucial p-chief factors. This fact leads to equivalent conditions on the $\mathfrak{F}$-projectors, the $\mathfrak{F}$-normalizers and the $\mathfrak{F}$-hypercenter.

Only finite groups will be considered. The terminology and notation are standard and may be found in Huppert [5] and Carter and Hawkes [1].

1. p-Locally defined formations

DEFINITION 1.1. Let G be a finite group and p a fixed prime number. Let $\Phi_p(G)$ denote the intersection of G with all maximal subgroups M of G such that $p \nmid [G:M]$. $\Phi_p(G)$ is called the p-Frattini subgroup of G.

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It follows directly from 1.1 that $\Phi_p(G)$ is a characteristic subgroup of G which contains the Frattini subgroup $\Phi(G)$ and that $\Phi_p(G) = G$ if and only if $G = 1$ or G is a p-group. The following lemma is easily verified.

LEMMA 1.2. *Let G be a finite group and $x \in G$. Then $x \in \Phi_p(G)$ if and only if when $G = \langle R, x\rangle$ and $p \nmid [G:\langle R\rangle]$, then $G = \langle R\rangle$.*

It is clear from 1.1 that P^*, the p-Sylow subgroup of $\Phi_p(G)$, is equal to $O_p(G)$ the largest normal p-subgroup of G. In [3, theorem 2] Deskins has shown that $\Phi_p(G)/P^* = \Phi_p(G/P^*) \neq \Phi(G/P^*)$. Consequently, $\Phi_p(G) = P^*N$ where N is a nilpotent group of p' order. Using this fact and 1.2 we prove the following lemmas.

LEMMA 1.3. *$\Phi_p(G) = P^*N$ is nilpotent if and only if $N \leqq \Phi(G)$.*

PROOF. If $\Phi_p(G)$ is nilpotent, $N \lhd G$. If $N \not\leqq \Phi(G)$, then $G = NM$ for some maximal subgroup M of G. But then $p \nmid [G:M]$ contradicting the fact that $N \leqq \Phi_p(G)$. Conversely, $N \leqq \Phi(G)$ implies that $\Phi_p(G) = P^*N \leqq P^*\Phi(G) \leqq F(G)$, the Fitting subgroup of G, and $\Phi_p(G)$ is nilpotent.

LEMMA 1.4. *$\Phi_p(G) = P^*N = \Phi(G)$ if and only if $P^* \leqq \Phi(G)$.*

PROOF. $P^* \leqq \Phi(G)$ implies that $\Phi_p(G)/\Phi(G)$ is nilpotent and therefore that $\Phi_p(G)$ is nilpotent. By 1.3, $N \leqq \Phi(G)$ and $\Phi_p(G) = \Phi(G)$. The converse is obviously true.

LEMMA 1.5. *Let q be a prime number different from p. Then $q \mid |G/\Phi_p(G)|$ if and only if $q \mid |G|$.*

PROOF. Suppose that $q \mid |G|$. If $q \nmid |\Phi_p(G)|$, then clearly $q \mid |G/\Phi_p(G)|$. If $q \mid |\Phi_p(G)|$, let P^* be the p-Sylow subgroup of $\Phi_p(G)$ and set $\bar{G} = G/P^*$. Then $q \mid |\Phi_p(\bar{G})| = |\Phi(\bar{G})|$. By [5, Satz 3.8 p. 270], $q \mid |\bar{G}/\Phi(\bar{G})| = |G/\Phi_p(G)|$.

That the condition $q \neq p$ is a necessary one in 1.5 is illustrated by S_3, the symmetric group on three symbols. $\Phi_3(S_3) \simeq C_3$ and $3 \mid |S_3|$, but $3 \nmid |S_3/\Phi_3(S_3)|$.

A formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$. Since $\Phi(G) \leqq \Phi_p(G)$, it is natural to ask what can be said about a formation $\mathfrak{F}$ such that $G/\Phi_{p(G)} \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$.

DEFINITION 1.6. A formation $\mathfrak{F}$ is called p-saturated if $G/\Phi_p(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$.

Of course, if $\mathfrak{F}$ is p-saturated, it must be a saturated formation containing all p-groups such that $\mathfrak{P}\mathfrak{F} = \mathfrak{F}$ where $\mathfrak{P}$ is the formation of p-groups.

The class of p-closed groups is an example of a p-saturated formation. Poljakov [6, corollary 4] has shown that the formation of q-nilpotent groups is also p-saturated if $q \neq p$. The class of nilpotent groups is a saturated formation which is not p-saturated for any prime number p.

DEFINITION 1.7. A formation $\mathfrak{F}$ is said to be p-locally defined if it is locally defined by a system $\{\mathfrak{F}(t)\}$ where $\mathfrak{F}(p) = \mathfrak{F}$.

Let $\mathfrak{F}$ be the formation of p-closed groups. Then $\mathfrak{F}$ may be p-locally defined by $\{\mathfrak{F}(t)\}$ where $\mathfrak{F}(t)$ is the formation of p'-groups for $t \neq p$ and $\mathfrak{F}(p) = \mathfrak{F}$. If q is a prime number different from p, the class of q-nilpotent groups and the class of q-solvable groups with q-length $l_q \leqq 1$ can also be p-locally defined.

The following result shows that p-locally defined formations bear the same relation to p-saturated formations as locally defined formations bear to saturated formations.

THEOREM 1.8. *Every p-locally defined formation is p-saturated.*

PROOF. Suppose not and let $\mathfrak{F} = \{\mathfrak{F}(t)\}$ be a p-locally defined formation. Let G be a group of minimal order such that $G/\Phi_p(G) \in \mathfrak{F}$ and $G \notin \mathfrak{F}$. Then $\Phi_p(G) \neq \Phi(G)$ and $O_p(G) \neq 1$. Let K be a minimal normal subgroup of G contained in $O_p(G)$ and set $\bar{G} = G/K$. Then $\bar{G}/\Phi_p(\bar{G}) \simeq G/\Phi_p(G) \in \mathfrak{F}$ and $G/K \in \mathfrak{F}$. Since $K \leqq C_G(K)$, $G/C_G(K) \in \mathfrak{F} = \mathfrak{F}(p)$ and $G \in \mathfrak{F}$. This contradiction establishes that $\mathfrak{F}$ is p-saturated.

It should be noted that if π, the support of $\mathfrak{F}$, is not the set of all prime numbers, one must verify that if $G/\Phi_p(G)$ is a π-group then G is also a π-group. But this is ensured by 1.5.

2. p-Saturated formations of solvable groups

A saturated formation $\mathfrak{F}$ consisting of only solvable groups can be locally defined by an integrated system $\{\mathfrak{F}(t)\}$, i.e., $\mathfrak{F}(t) \leqq \mathfrak{F}$ for every prime number t. If the group G is solvable every maximal subgroup will have prime power index in G and G will have an $\mathfrak{F}$-projector ($\mathfrak{F}$-covering subgroup), an $\mathfrak{F}$-normalizer and an $\mathfrak{F}$-hypercenter. We want to relate these concepts to those of p-saturation and p-local definition.

Let $\mathfrak{F}$ be a p-locally defined formation of solvable groups with support π. Then $p \in \pi$ and even if π does not consist of all prime numbers, we may use the results of Carter and Hawkes [1] by adopting the definition of $\mathfrak{F}$-normalizer given by Seitz and Wright [7] and making other minor adjustments.

DEFINITION 2.1. Let t be a prime number and M a maximal subgroup of G such that $[G:M]=t^{\alpha}$. M is called a t-maximal subgroup of G. M is called $\mathfrak{F}$-normal if $t\in\pi$ and $M/\mathrm{core}_G M\in\mathfrak{F}(t)$. Otherwise M is called $\mathfrak{F}$-abnormal. If M is $\mathfrak{F}$-abnormal, M is said to be $\mathfrak{F}$-critical if $t\notin\pi$ or if $G=F(G)M$. An $\mathfrak{F}$-abnormal maximal subgroup M is said to be $\mathfrak{F}$-crucial if $t\notin\pi$ or if $M/\mathrm{core}_G M\in\mathfrak{F}$.

As in [1], $G\in\mathfrak{F}$ if and only if G has no $\mathfrak{F}$-abnormal maximal subgroups. Similarly, $G\in\mathfrak{F}$ if and only if G has no $\mathfrak{F}$-critical maximal subgroups and if and only if G has no $\mathfrak{F}$-crucial maximal subgroups.

We will make use of the following results from [1].

LEMMA 2.2 [1, lemma 4.6]. *If M is an $\mathfrak{F}$-critical maximal subgroup of G then each $\mathfrak{F}$-normalizer of M is an $\mathfrak{F}$-normalizer of G.*

LEMMA 2.3 [1, theorem 4.7]. *Each $\mathfrak{F}$-normalizer D of G can be joined to G by a chain of the form*

$$D=G_r\leqq G_{r-1}\leqq\cdots\leqq G_1\leqq G_0=G$$

where G_i is an $\mathfrak{F}$-critical maximal subgroup of G_{i-1}, $1\leqq i\leqq r$.

LEMMA 2.4 [1, theorem 5.4]. *Each $\mathfrak{F}$-projector F of G can be joined to G by a chain of the form*

$$F=G_n\leqq G_{n-1}\leqq\cdots\leqq G_1\leqq G_0=G$$

where G_i is an $\mathfrak{F}$-crucial maximal subgroup of G_{i-1}, $i=1,\cdots,n$.

THEOREM 2.5. *Let $\mathfrak{F}$ be a saturated formation of solvable groups. Then the following statements are equivalent:*

(a) *$\mathfrak{F}$ is p-saturated.*
(b) *$\mathfrak{F}$ can be p-locally defined by an integrated system $\{\mathfrak{F}(t)\}$ with $\mathfrak{F}(p)=\mathfrak{F}$.*
(c) *Every solvable group G has an $\mathfrak{F}$-projector F such that $p\nmid[G:F]$.*
(d) *Every solvable group G has an $\mathfrak{F}$-normalizer D such that $p\nmid[G:D]$.*
(e) *For every solvable group G, $O_p(G)\leqq Z_{\mathfrak{F}}(G)$.*

PROOF. 1. (a) implies (b). $\mathfrak{F}$ is saturated so there is an integrated system $\{\mathfrak{F}(t)\}$ of formations which locally define $\mathfrak{F}$. Let $\mathfrak{F}^*$ be the saturated formation locally defined by $\{\mathfrak{F}^*(t)\}$ where $\mathfrak{F}^*(t)=\mathfrak{F}(t)$ if $t\neq p$ and $\mathfrak{F}^*(p)=\mathfrak{F}$. It is clear that $\mathfrak{F}\leqq\mathfrak{F}^*$ and the two formations have the same support. Suppose that $\mathfrak{F}\neq\mathfrak{F}^*$. Let G be a solvable group of minimal order such that $G\in\mathfrak{F}^*$ but $G\notin\mathfrak{F}$ and let N be minimal normal subgroup of G. Then $G/N\in\mathfrak{F}$ and N must be the unique minimal normal subgroup of G. $G \notin \mathfrak{F}$ implies that $G/\Phi_p(G) \notin \mathfrak{F}$, so $\Phi_p(G) = 1$ and G has no normal p-subgroups. Hence $|N| = q^r$ for some prime number $q \neq p$. $G \in \mathfrak{F}^*$, so every t-chief factor of G is $\mathfrak{F}$-central if $t \neq p$. Let H/K be a p-chief factor of G. Then H/K must lie above N and is thus G-isomorphic to a p-chief factor of G/N. Since $G/N \in \mathfrak{F}$, H/K is $\mathfrak{F}$-central in G and $G \in \mathfrak{F}$. But this is not possible, so $\mathfrak{F} = \mathfrak{F}^*$.

2. (b) implies (c). It is clear from Definition 2.1 that a solvable group has no $\mathfrak{F}$-crucial p-maximal subgroups. If F is an $\mathfrak{F}$-projector of G, let

$$F = G_n < G_{n-1} < \cdots < G_1 < G_0 = G$$

be an $\mathfrak{F}$-crucial chain. Since $p \nmid [G_{i-1} : G_i]$ for $i = 1, 2, \cdots, n$, $p \nmid [G : F]$.

3. (c) implies (d). We show first that if D is an $\mathfrak{F}$-normalizer of G, then $O_p(G) \leqq D$. Suppose not and let G be a minimal counter-example. Then $G \notin \mathfrak{F}$ and $O_p(G) \neq 1$. Let K be a minimal normal subgroup of G contained in $O_p(G)$. If $K \leqq M$ where M is an $\mathfrak{F}$-critical maximal subgroup of G, $K \leqq O_p(M) \leqq D$. Then $O_p(G)/K = O_p(G/K) \leqq D/K$ and $O_p(G) \leqq D$. Hence K is not contained in any $\mathfrak{F}$-critical maximal subgroup of G. So for any $\mathfrak{F}$-critical maximal subgroup M of G we have $G = KM$ and $M \simeq G/K \in \mathfrak{F}$. But this implies that M is an $\mathfrak{F}$-projector of G, contradicting the fact that $p \mid [G : M]$. This contradiction establishes that $O_p(G) \leqq D$.

Now let H/K be a p-chief factor of G. $H/K \leqq O_p(G/K) \leqq DK/K$ so that H/K, since it is covered by D, is $\mathfrak{F}$-central in G. Since every p-chief factor of G is $\mathfrak{F}$-central, $p \nmid [G : D]$.

4. (d) implies (e). Since each $\mathfrak{F}$-normalizer of G contains a p-Sylow subgroup of G, $O_p(G) \leqq Z_{\mathfrak{F}}(G)$, the intersection of the $\mathfrak{F}$-normalizers.

5. (e) implies (a). Let $G/\Phi_p(G) \in \mathfrak{F}$. If $\Phi_p(G) = \Phi(G)$, $G \in \mathfrak{F}$ since $\mathfrak{F}$ is saturated. If $\Phi_p(G) \neq \Phi(G)$, 1.4 implies that $O_p(G) \neq 1$. Let $\bar{G} = G/O_p(G)$. Then $\bar{G}/\Phi(\bar{G}) \simeq G/\Phi_p(G) \in \mathfrak{F}$, so $\bar{G} = G/O_p(G) \in \mathfrak{F}$. Since $O_p(G) \leqq Z_{\mathfrak{F}}(G)$, $G \in \mathfrak{F}$ and $\mathfrak{F}$ is p-saturated.

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